Geometric Phase in Eigenspace Evolution of Invariant and Adiabatic Action Operators

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The theory of geometric phase is generalized to a cyclic evolution of the *eigenspace* of an invariant operator with N-fold degeneracy. The corresponding geometric phase is interpreted as a holonomy inherited from the universal connection of a Stiefel $U(N)$-bundle over a Grassmann manifold. Most significantly, for an arbitrary initial state, this geometric phase captures the inherent geometric feature of the state evolution. Moreover, the geometric phase in the evolution of the eigenspace of an adiabatic action operator is also addressed, which is elaborated by a pullback $U(N)$-bundle.

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Quantum geometric phases have been attracting significant research interests, and have found many applications in various fields including potential energy surfaces, the molecular Aharonov-Bohm effect, Bloch bands in condense matters, and quantum Hall effects *etc.* [1, 2]. A pioneering theory of geometric phase was established by Berry for an adiabatic cyclic evolution of a non-degenerate energy eigenstate [3], and its holonomy interpretation was given by Simon [4]. Latter, the geometric phase was respectively generalized to the cases in the adiabatic evolution of degenerate energy eigenstates by Wilczek and Zee [5], and in the evolution of a cyclic quantum state by Aharonov and Anandan (AA) [6]; the potential applications were addressed even in quantum computation [7]. Very recently, an important attempt was made to introduce phenomenologically a geometric phase accumulated in the evolution that preserves a classical action adiabatically [8]. Thus it is fundamental and significant to generalize the geometric phase in the evolution of eigenspaces of an adiabatic action operator, and particularly in the evolution of degenerate eigenspaces of an invariant operator in a system with a periodic time-dependent Hamiltonian, where even if the initial state is an eigenstate of the invariant operator this state may not evolve cyclically and thus the existing AA-phase theory is not applicable. In this paper, we establish a rigorous theory for the geometric phase in the cyclic evolution of degenerate eigenspaces of an invariant operator, and provide a holonomy interpretation inherited from the universal connection of a Stiefel $U(N)$-bundle over a Grassmann manifold. Most importantly, for an arbitrary initial state, this geometric phase captures the inherent geometric feature of the state evolution. Moreover, we also describe the geometric phase in a cyclic evolution of the eigenspace of an adiabatic action operator.

Geometric Berry and AA Phases The Hamiltonian $\hat{H}(R)$ of a quantum system considered by Berry[3] and Simon[4] is a linear Hermitian operator on a complex Hilbert space $\mathbf{H}$, and depends smoothly on an external parameter R in a parameter space M, which is a smooth manifold. If the system is non-degenerate, we have a complete set of normalized eigenstates $|n;R\rangle$, with the energy eigenvalue $E_n(R)$ being a smooth function defined globally on M. On the other hand, there are various choices of $|n;R\rangle$. The dependence of an n^{th}-level eigenstate on the parameter R is referred as an *interpretation*. Clearly, one may define a new *interpretation* $|n;R\rangle'$ by a Gauge transformation. We here are interested in single-valued *interpretations* that depend smoothly on a local domain in the parameter space. Consider the evolution of a physical *pure* state $|\psi(t)\rangle\langle\psi(t)|$, where the state ket $|\psi(t)\rangle$ is governed by the Schrödinger's equation, with a time dependent parameter $R(t)$. For a cyclic evolution C in the parameter R, i.e, $R(0){=}R(T)$, the adiabatic approximation [3] with the initial condition $|\psi(0)\rangle = |n;R(0)\rangle$ leads to the solution $|\psi(t)\rangle \approx k_n(t)\,|n;R(t)\rangle$ with $k_n(T) = \exp\left(iD_n(T) + i\gamma_n(C)\right)$, where $D_n(T) = -\int_0^T E_n(R(t))/\hbar dt$ and $\gamma_n(C) = i\int_C \langle n;R|\, d\,/dR\,|n;R\rangle\, dR$ are the dynamic phase and geometric Berry phase respectively [3].

From any arbitrary local *interpretation*, one can construct a trivial $U(1)$-bundle over the domain of the *interpretation* in M, where the fibre of each R is $U(1)\otimes|n;R\rangle$. Suppose that two *interpretations* are related by the Gauge transformation $g_{\alpha\beta}$ with $|n;R\rangle_\beta = g_{\alpha\beta}(R)\,|n;R\rangle_\alpha$, which gives a way to identify points between trivial bundles defined by the *interpretations*. By attaching these local bundles together, we obtain a principal $U(1)$-bundle [9, 10] P over M. Equivalently, we can also construct a complex line bundle L over M to describe the system, where the fibre space of each R is given by $\mathbb{C}\otimes|n;R\rangle$. This line bundle is the bundle induced from P by the representation of the natural inclusion $i : U(1) \hookrightarrow \mathbb{C}$ [10]. The geometry of the bundle is described by the Berry-Mead connection written in the form of a set of local connection 1-forms:

$$\theta_\alpha = \langle n;R|\, d\,|n;R\rangle_\alpha\,. \tag{1}$$

This connection gives a horizontal lifting of a curve in M. The lifting is just the evolution of the state without the dynamic phase. If the evolution is cyclic, the Berry phase is interpreted as a holonomy in the principal $U(1)$-bundle (or complex line bundle) [9].

For the non-adiabatic but cyclic evolutions of physical *pure* states, i.e. $|\psi(0)\rangle\langle\psi(0)| = |\psi(T)\rangle\langle\psi(T)|$ (or $|\psi(T)\rangle = \exp(i\gamma(T))|\psi(0)\rangle$ with a real γ as the total phase shift), Aharonov and Anandan [6] considered the space of *pure* states as a complex projective Hilbert space $P(\mathbf{H})$. Choosing a time-dependent normalized ket $|\phi(t)\rangle$ such that $|\psi(t)\rangle\langle\psi(t)| = |\phi(t)\rangle\langle\phi(t)|$ and $|\psi(0)\rangle = |\phi(0)\rangle = |\phi(T)\rangle$, although $|\phi(t)\rangle$ may not be a solution of the Schrödinger's equation, it helps to give a key result: $\gamma(T) = D(T) + \gamma_{AA}(C)$, where $D(T) = -\int_0^T \langle\phi(t)|\hat{H}(t)|\phi(t)\rangle dt/\hbar$ and $\gamma_{AA}(C) = i\int_C \langle\phi(t)|d/dt|\phi(t)\rangle\, dt$ are the dynamic phase and geometric AA phase, respectively.

The geometric picture of AA phase is the AA bundle η, also known as the Hopf bundle [10], which is a principal $U(1)$-bundle over the projective Hilbert space $P(\mathbf{H})$. This bundle is the universal $U(1)$-bundle [11]. An equivalent description is the canonical line bundle [10] ξ over the projective Hilbert space induced from the Hopf bundle by using the representation of inclusion $i : U(1) \hookrightarrow \mathbb{C}$. The geometry of this bundle is given by the local connection 1-form known as the universal connection [12, 13] of the universal bundle:

$$\omega_\alpha = \langle\phi_\alpha|\, d\, |\phi_\alpha\rangle\,. \tag{2}$$

Now, the relationship between the geometric pictures of Berry phase and AA phase [12] becomes clear. Although the *interpretation* of energy eigenstate $|n;R\rangle$ is local, the function $f : M \to P(\mathbf{H})$ can be extended globally on the parameter space M, mapping $R \mapsto |n;R\rangle\langle n;R|$. Obviously, this map is differentiable, and the principal $U(1)$-bundle P (the induced line bundle L) is the pullback bundle [10] of the Hopf bundle η (the canonical line bundle ξ) by f, i.e. $P = f^{<}\eta$ ($L = f^{<}\xi$), where $f^{<}$ denotes the covariant pullback functor of bundles induced from f. The Berry-Mead connection is the pullback of the universal connection $\theta = f^{<}\omega$ [10].

Invariant Operators and Related Geometric Phases If a Hermitian operator $\hat{I}(t)$ depends smoothly on time and satisfies

$$\frac{\partial \hat{I}}{\partial t} - \frac{i}{\hbar}[\hat{I},\hat{H}] \equiv 0, \tag{3}$$

it is an invariant operator of the system [14]. Intriguing properties of an invariant operator include: (i) all eigenvalues are time-independent; (ii) if the state $|\psi_0\rangle$ is an eigenstate of $\hat{I}(t_0)$, its evolution $|I(t)\rangle = U(t;t_0)|\psi_0\rangle$ is always an eigenstate of $\hat{I}(t)$ with the same eigenvalue, where $U(t;t_0)$ is the time-evolving operator; and (iii) transitions among eigenspaces specified by different eigenvalues are impossible. We now first consider the non-degenerate case of the invariant operator. Whenever the following periodic condition is satisfied,

$$\hat{I}(0) = \hat{I}(T), \hat{H}(0) = \hat{H}(T) \tag{4}$$

with at least one of $\hat{I}$ and $\hat{H}$ being time-dependent, the evolution of its eigenstate is found to be cyclic. Thus we can deal with the geometric phase in the evolution of an eigenspace of the invariant operator in the framework of the AA phase theory. In the present case, each eigenspace of the invariant operator is a ray in the Hilbert space and hence is a point in the projective Hilbert space, so that the evolution of eigenspaces corresponds exactly to the evolution of physical *pure* states.

As a simple but interesting example, let us consider an electron subject to a time-independent magnetic field. The Hamiltonian reads $\hat{H} = \mu_B B\sigma_z$ with eigenstates $|\xi_\pm\rangle$ and eigen-energies $\pm\mu_B B$. Denoting the initial state of the system as one of the two orthonormal states: $|\psi_\pm(0)\rangle = \pm\cos\theta|\xi_\pm\rangle + \sin\theta|\xi_\mp\rangle$, we have $|\psi_\pm(t)\rangle = e^{\pm i\omega_s t/2}|\phi_\pm\rangle$, where

$$|\phi_\pm\rangle = \pm\cos\theta|\xi_\pm\rangle + e^{\pm i\omega_s t}\sin\theta|\xi_\mp\rangle\,, \tag{5}$$

and $\omega_s = 2\mu_B B/\hbar$. Then, an invariant operator is found to be $\hat{I}(t) = |\phi_+\rangle\langle\phi_+| - |\phi_-\rangle\langle\phi_-| = \sin 2\theta\cos\omega_s t\sigma_x + \sin 2\theta\sin\omega_s t\sigma_y + \cos 2\theta\sigma_z$, where $|\phi_\pm\rangle$ are its eigenstates with eigenvalues ± 1 and σ_α's are the Pauli matrices. The connection 1-forms are $\omega_\pm(t) = \langle\phi_\pm|d/dt|\phi_\pm\rangle\, dt = \pm i\omega_s(1-\cos 2\theta)dt/2$, and the geometric phases for one cyclic evolution are $\gamma_\pm = i\int_0^T \omega_\pm = \pi(1\pm\cos 2\theta)$.

As a more complicated example, we consider an electron in a one-dimensional ring with radius a and subject to a crown-shaped electric field $\mathbf{E}(\varphi) = E\sin\chi\cos\varphi\hat{e}_x + \sin\chi\sin\varphi\hat{e}_y + \cos\chi\hat{e}_z$ [16]. The Hamiltonian reads

$$\hat{H} = \hbar\Omega\left[-i\frac{\partial}{\partial\varphi} + \hat{s}_\varphi\right]^2, \tag{6}$$

where $\hat{s}_\varphi = -\epsilon/2(\cos\chi\cos\varphi\sigma_x + \cos\chi\sin\varphi\sigma_y - \sin\chi\sigma_z)$, $\epsilon = \mu_B Ea/\hbar$, and $\Omega = \hbar/2ma^2$. The corresponding non-degenerate energy eigenstates are

$$\begin{aligned} |\xi_{n+}\rangle &= e^{in\varphi}\begin{pmatrix}\cos\Theta \\ e^{i\varphi}\sin\Theta\end{pmatrix}, \\ |\xi_{n-}\rangle &= e^{in\varphi}\begin{pmatrix}-\sin\Theta \\ e^{i\varphi}\cos\Theta\end{pmatrix}, \end{aligned} \tag{7}$$

where $\tan 2\Theta = \Delta/g$ with $\Delta = \epsilon\cos\chi$ and $g = 1-\epsilon\sin\chi$. Letting $|\psi_{n\pm}(0)\rangle = \pm\cos\theta_n|\xi_{n\pm}\rangle + \sin\theta_n|\xi_{n\mp}\rangle$, since the Hamiltonian is time-independent, we can use the same scenario to obtain the invariant operator

$$\hat{I}(t) = \sum_n \left[(I_n+1)|\phi_{n+}\rangle\langle\phi_{n+}| + (I_n-1)|\phi_{n-}\rangle\langle\phi_{n-}|\right], \tag{8}$$

with its eigenstates being

$$|\phi_{n\pm}\rangle = \pm\cos\theta_n|\xi_{n\pm}\rangle + e^{\pm i\Omega_{ns}t}\sin\theta_n|\xi_{n\mp}\rangle, \tag{9}$$

where $\Omega_{ns} = \Omega(n+1/2)\sqrt{\Delta^2+g^2}$ is the angular frequency for spin precession, and I_n can be chosen to be an arbitrary real number such that the eigenvalues $I_n \pm 1$ are non-degenerate. The connection 1-forms are $\omega_{n\pm}(t) = \pm i\Omega_{ns}(1-\cos 2\theta_n)dt/2$, and the geometric phases for one cyclic evolution of the eigenstates are $\gamma_{n\pm} = \pi(1\pm\cos 2\theta_n)$.

At this stage, it is more significant and highly nontrivial to tackle a general degenerate case of the invariant operator because a state may not evolve cyclically even if it is initially an eigenstate of $\hat{I}$ and Eq.(4) is satisfied. In this case, supposing that the n-th level of the invariant operator has an N-fold degeneracy, the n-th eigenspace has a complex dimension larger than one. Hence, the evolution of this eigenspace can no longer be viewed as the evolution of a physical *pure* state in the projective Hilbert space, because the eigenspace is not a ray but a higher dimensional subspace in the Hilbert space. Indeed, when Eq.(4) is satisfied, each eigenspace does undergo a cyclic evolution, rather than the eigenstate. Hereafter, we define this kind of cyclic evolution of the eigenspace of the invariant operator as the non-Abelian cyclic evolution, with Eq.(4) being its condition. A proper geometric object tackling an N-fold degeneracy is the complex Grassmann manifold $Gr(K;N)$ [10], where $Gr(K;N)$ is the space of N-dimension subspaces in the Hilbert space with K as the complex dimension of the Hilbert space [17]. Based on this picture, the evolution of the eigenspace can be treated as a curve in $Gr(K;N)$, and the evolution is non-Abelian cyclic only if the curve is a loop. The corresponding geometric phase is non-Abelian. In fact, as addressed in Ref.[12], the non-Abelian geometric phase is interpreted as a holonomy of the Stiefel $U(N)$-bundle $V_0(K;N)$[10], with $V_0(K;N)$ being the space of orthogonal N-frames of the Hilbert space, over the Grassmann manifold $Gr(K;N)$ with a universal connection [12, 13], which is written as a local connection 1-form $\omega_\alpha = ({\omega_\alpha}^r_s)_{N\times N}$ with value inside the Lie algebra $u(N)$ of skew-Hermitian matrices:

$$ {\omega_\alpha}^r_s = \langle \phi_{\alpha r} | \, d \, | \phi_{\alpha s} \rangle , \tag{10} $$

where ϕ_α is a local *section* [9, 10] from the Grassmann manifold to the Stiefel bundle, mapping V to $(\phi_{\alpha 1}(V), \ldots, \phi_{\alpha N}(V))$ with $(\phi_{\alpha 1}(V), \ldots, \phi_{\alpha N}(V))$ as an orthonormal N-frame that spans the space V.

An equivalent method to address this issue is to use the canonical holomorphic vector bundle Ξ [10] of rank N over the Grassmann manifold with the universal connection described by Eq.(10). This method seems more physical, since the fibre space over a point V in the Grassmann manifold can be treated as the vector space V itself. Upon the evolution of the invariant operator, a curve is parameterized on the Grassmann manifold and the fibre space over each point on the curve can be viewed as the eigenspace of the operator.

Let the non-Abelian cyclic evolution define a loop $t \mapsto V(t)$ in $Gr(K;N)$ that can be covered by the domain of one local *section* ϕ_α. This *section* can then be treated as a curve in the canonical vector bundle parameterized by time in the following composition $\phi_\alpha : t \mapsto V(t) \mapsto \phi_\alpha(V(t))$, with $V(0) = V(T)$ and $\phi_\alpha(0) = \phi_\alpha(T)$. The corresponding geometric phase is

$$ \Gamma(C) = i \int_C \omega_\alpha , \tag{11} $$

where ω_α is the 1-form in Eq.(10). Remarkably, this geometric phase captures the inherent geometric feature of the related evolution, even for an *arbitrary* initial state. This is because the state evolution can be decomposed into linear combinations of the invariant operator eigenstates with time-independent expansion-coefficients according to the property (ii) below Eq.(2) and thus the geometric phase (matrix) can be represented by the direct sum $\Gamma = \oplus_n \Gamma_n$ with Γ_n as the geometric phase for the n-th eigenspace. Therefore, the present invariant operator scenario exhibits a superior advantage in exploring the geometric nature of the non-Abelian cyclic evolution of the system with an arbitrary initial state [15].

As an intriguing illustration, let us still consider an electron in a ring, but subject to a rotating electric field $\mathbf{E}(\varphi,t) = E \sin\chi \cos(\varphi - \omega_o t)\hat{e}_x + \sin\chi \sin(\varphi - \omega_o t)\hat{e}_y + \cos\chi \hat{e}_z$ with a non-zero ω_o. The Hamiltonian Eq.(6) will be time dependent by replacing $\hat{s}_\varphi$ with $\hat{s}_{\varphi-\omega_o t}$. The two n-th level degenerate eigenstates of the invariant operator $\hat{J} = -i\frac{\partial}{\partial\varphi} + \frac{1}{2}\sigma_z$ with eigenvalue $n + 1/2$ are

$$ \begin{aligned} |\xi'_{n+}\rangle &= e^{in\varphi} \begin{pmatrix} \cos\Theta \\ e^{i(\varphi-\omega_o t)} \sin\Theta \end{pmatrix}, \\ |\xi'_{n-}\rangle &= e^{in\varphi} \begin{pmatrix} -\sin\Theta \\ e^{i(\varphi-\omega_o t)} \cos\Theta \end{pmatrix}. \end{aligned} \tag{12} $$

Using Eqs. (10)-(12), the non-Abelian geometric phase is found to be the Hermitian matrix

$$ \Gamma_n = \begin{pmatrix} \pi(1-\cos 2\Theta) & \pi \sin 2\Theta \\ \pi \sin 2\Theta & \pi(1+\cos 2\Theta) \end{pmatrix}. \tag{13} $$

The matrix is independent of n and has both diagonal and off-diagonal terms, reflecting the spin precession and flipping in the non-Abelian cyclic evolution.

Adiabatic Action Operators and Geometric Phases A linear Hermitian operator $\hat{A}$ is defined to be an action operator if it commutes with the Hamiltonian. Here the action operator $\hat{A}(R)$ and the Hamiltonian $\hat{H}(R)$ are assumed to depend smoothly on an external parameter R in a parameter space M, as in the Berry phase case. When we consider the evolution of the external parameter $R(t)$ and if $\partial\hat{A}(R(t))/\partial t \approx 0$, the action operator $\hat{A}$ is said to be approximately adiabatic in the evolution. Since $[\hat{A}(R), \hat{H}(R)] = 0$, $\hat{A}$ satisfies Eq.(3) approximately, namely it is an approximate invariant operator.

The geometric phase of an eigenspace of the adiabatically evolving action operator can be elaborated as follows. By using the smooth function $f : M \to Gr(K;N)$, where $f(R)$ is the eigenspace of the action operator $\hat{A}(R)$, we can define the pullback principal bundle $P = f^{<} V_0(K;N)$ (or the vector bundle $\Pi = f^{<}\Xi$), described by the local pullback connection 1-form $\theta_\alpha = f^{<}\omega_\alpha$.

To evaluate the geometric phase, we may choose an arbitrary smooth local *interpretation* $R \mapsto |m, r; R\rangle_\alpha$ satisfying $f_{>} |m, r; R\rangle_\alpha = \phi_{\alpha r}(f(R))$, where $f_{>} : \Pi \to \Xi$ is the pushforward between bundles [10], m denotes the eigenlevel of the action operator, r runs from one to N.

Hence, the set of states $\{|m,r;R\rangle_\alpha\}_\alpha$ form an orthonormal basis of the eigenspace $f(R)$. Consequently, we can write the connection for the bundles P (or Π) explicitly as $\theta_\alpha = f^{<}\omega_\alpha = (\theta_{\alpha}{}_s^r)_{N\times N}$, where

$$\theta_{\alpha}{}_s^r = \langle m,r;R|\, d\, |m,s;R\rangle_\alpha\,. \tag{14}$$

The geometric phase is then

$$\Gamma(C) = i\int_C \theta_\alpha. \tag{15}$$

Interestingly, according to the Liouville's theorem in classical mechanics [18], the symplectic manifold representing a periodic integrable system can be decomposed into leaves such that each leaf is a submanifold diffeomorphic to a generalized torus, and each leaf possesses a set of constants of motion: $\boldsymbol{I} = \{I^1,\ldots,\ I^n\}$ with their conjugate variables being angles $\varphi = \{\varphi^1,\ldots,\ \varphi^n\}$, where φ parameterize the n-dimensional torus T^n. For the corresponding quantum system, let the wave function $|\psi(\boldsymbol{I})\rangle = \psi(\boldsymbol{I};\varphi)$ be an eigenfunction of an action operator $\hat{A}$ with eigenvalues $\boldsymbol{I}$. The inner product is defined as $\langle\psi(\boldsymbol{I}) \mid \phi(\boldsymbol{I})\rangle = \oint_{T^n}\psi(\boldsymbol{I};\varphi)^\dagger\phi(\boldsymbol{I};\varphi)d\varphi/(2\pi)^n$. We now consider a non-degenerate case of the adiabatic action operator depending on a slowly evolving external parameter R. For a cyclic evolution C of the parameter, from Eq.(15), the geometric phase is a real number [19]:

$$\gamma(C) = \frac{1}{(2\pi)^n}\oint_{T^n}\oint_C \psi\left(\boldsymbol{I}(R);\varphi\right)^\dagger \frac{\partial\psi}{\partial R}(\boldsymbol{I}(R);\varphi)dRd\varphi. \tag{16}$$

This is just the geometric phase introduced phenomenologically in Eq.(2) of Ref.[8]. We here present a rigorous derivation with a precise geometric interpretation.

To illustrate an application of Eq.(16), we consider the slowly evolving system described above Eq.(12) with a very small non-zero ω_0. The external parameter is now the angle $\vartheta = \omega_0 t\ (mod\ 2\pi)$, $\mathbf{E}(\varphi,t) = E\sin\chi\cos(\varphi-\vartheta)\hat{e}_x + \sin\chi\sin(\varphi-\vartheta)\hat{e}_y + \cos\chi\hat{e}_z$, and the parameter space of ϑ is the unit circle S^1. The Hamiltonian is dependent on ϑ by replacing $\hat{s}_\varphi$ by $\hat{s}_{\varphi-\vartheta}$ in Eq.(6). A natural choice of an action operator is $\hat{A}(\vartheta) = -i\frac{\partial}{\partial\varphi} + \hat{s}_{\varphi-\vartheta}$. The two eigenstates of the action operator are

$$\begin{aligned}\left|\xi'_{n+}(\vartheta)\right\rangle &= \xi'_{n+}(\vartheta,\varphi) = e^{in\varphi}\begin{pmatrix}\cos\Theta\\ e^{i(\varphi-\vartheta)}\sin\Theta\end{pmatrix},\\ \left|\xi'_{n-}(\vartheta)\right\rangle &= \xi'_{n-}(\vartheta,\varphi) = e^{in\varphi}\begin{pmatrix}-\sin\Theta\\ e^{i(\varphi-\vartheta)}\cos\Theta\end{pmatrix}\end{aligned} \tag{17}$$

with eigenvalues $n + (1\pm\sqrt{\Delta^2+g^2})/2$. The connection 1-forms are $\theta_{n\pm}(\vartheta) = \left\langle\xi'_{n\pm}(\vartheta)\right|\frac{d}{d\vartheta}\left|\xi'_{n\pm}(\vartheta)\right\rangle d\vartheta = -i(1\pm\cos 2\Theta)d\vartheta/2$. Using Eq.(16), the geometric phases of one cyclic evolution of ϑ in the two eigenstates are found to be

$$\gamma_{n\pm} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\xi'^{\,*}_{n\pm}\frac{\partial\xi'_{n\pm}}{\partial\vartheta}d\vartheta d\varphi = \pi(1\mp\cos 2\Theta). \tag{18}$$

Under this adiabatic approximation, the spin flipping is neglected.

Summary We have developed a general theory of geometric phase in the eigenspace evolution of invariant and adiabatic action operators and elaborated its holonomy interpretation based on the fiber bundle theory. Our theory has been applied to non-trivial non-Abelian cases as well as to approximately adiabatic action evolutions successfully. We anticipate that the present theory will have more applications mainly because it is applicable for any initial state of the system satisfying Eq.(4) .

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[19] Eq.(16) is proven (not shown here) to be also valid for a non-linear $U(1)$-invariant Hamiltonian(See, e.g., Ref.[8]).